Exercises for Stochastic Processes

Tutorial exercises:

T1. (a) Show that τ is a stopping time if and only if for all $t \geq 0$ it holds that $\{\tau < t\} \in \mathcal{F}_t$.

(b) Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times. Show that $\sup_n \tau_n$, $\inf_n \tau_n$, $\limsup_{n\to\infty} \tau_n$, $\liminf_{n\to\infty} \tau_n$ and, if existent, $\lim_{n\to\infty} \tau_n$ are stopping times.

T2. Let τ and $(\tau_n)_{n\in\mathbb{N}}$ be stopping times with respect to the right-continuous filtration $(\mathcal{F}_t)$ associated to Brownian motion.

(a) Show that

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \mid \forall t \geq 0 : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$$

is a σ-algebra.

(b) Show that τ is $\mathcal{F}_\tau$-measurable.

(c) Show that, if $\tau_1 \leq \tau_2$, then $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$.

(d) Show that, if $\tau_n \downarrow \tau$, then $\mathcal{F}_\tau = \bigcap_n \mathcal{F}_{\tau_n}$.

T3. Determine the distribution of

(a) $\tau_1 := \inf\{t \geq 1 \mid B_t = 0\}$ and

(b) $\tau_2 := \sup\{t < 1 \mid B_t = 0\}$,

for a Brownian motion B starting in the origin.

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Homework exercises:

H1. Show that, if B is a Brownian motion and τ a finite stopping time (with respect to the corresponding right-continuous filtration $(\mathcal{F}_t)$), then $Y_t := B_{\tau+t} - B_\tau$ defines a Brownian motion, which is independent of $\mathcal{F}_\tau$.

H2. The "tail σ-algebra" with respect to Brownian motion $B_t(\omega) = \omega(t)$ on $C[0,\infty)$ is defined as

$$\mathcal{T} := \bigcap_{t>0} \sigma\big(\{B_s \mid s \geq t\}\big) .$$

(a) Show that, for any $A \in \mathcal{T}$, $\mathbb{P}^x(A) \in \{0,1\}$.

(b) Show that $\mathbb{P}^x(A)$ does not depend on x.

H3. Show that the (ω-dependent) set of times at which a Brownian motion has local maxima is a.s. dense in $[0,\infty)$.

H4. (a) Let B be a Brownian motion starting in the origin, $a > 0$ and

$$\tau_a := \inf\{t > 0 \mid B_t - t = a\} .$$

Show that, for $a, b > 0$,

$$\mathbb{P}(\tau_{a+b} < \infty \mid \tau_a < \infty) = \mathbb{P}(\tau_b < \infty).$$

(b) Conclude that $\sup_{t\geq 0}(B_t - t)$ has an exponential distribution.

Deadline: Monday, 11.11.19